Journal of Global Optimization **24:** 205–217, 2002.

An Adaptive Dual Parametrization Algorithm for Quadratic Semi-infinite Programming Problems*

Y. LIU and K.L. TEO
Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong

Abstract. The so called dual parametrization method for quadratic semi-infinite programming (SIP) problems is developed recently for quadratic SIP problems with a single infinite constraint. A dual parametrization algorithm is also proposed for numerical solution of such problems. In this paper, we consider quadratic SIP problems with positive definite objective and multiple linear infinite constraints. All the infinite constraints are supposed to be continuously dependent on their index variable on a compact set which is defined by a number equality and inequalities. We prove that in the multiple infinite constraint case, the minimu parametrization number, just as in the single infinite constraint case, is less or equal to the dimension of the SIP problem. Furthermore, we propose an adaptive dual parametrization algorithm with convergence result. Compared with the previous dual parametrization algorithm, the adaptive algorithm solves subproblems with much smaller number of constraints. The efficiency of the new algorithm is shown by solving a number of numerical examples.

Key words: Semi-infinite programming; Global optimization; Dual parametrization; Adaptive algorithm; Convergence

1. Introduction

Consider the following semi-infinite programming (SIP) problem

PROBLEM (P).

$$\min \quad f(x) = (1/2)x^TQx + p^Tx\,, \tag{1}$$

$$\text{subject to} \quad A(y)x - b(y) \leqslant 0 \quad \text{for } y \in Y\,, \tag{2}$$

where $x = (x_1, x_2, \dots, x_n)^T \in R^n$ is the decision vector, $p = (p_1, p_2, \dots, p_n)^T \in R^n$ is a constant vector, $Q \in R^{n\times n}$ is a positive definite matrix, and $A(y)$: $y \to R^{m\times n}$ and $b(y)$: $y \to R^m$ are continuously differentiable functions defined on a given compact set $Y \subset R^s$. The vector inequality (2) is to be understood as component-wise inequalities.

Dual parametrization method was developed recently for the case $m = 1$ [6,7]. It is shown in [7] that for the case $m = 1$, of all the infinitely many constraints, only no

* This research is partially supported by the Research Committee of the Hong Kong Polytechnic University and the Research Grant Committee of Hong Kong (Grant Number: PolyU 5101/01E).

more than n of them will be sufficient for locating the optimal solution. In other words, for some $k \leq n$, the solution of the SIP problem is the same as the solution of a finite quadratic programming problem obtained by replacing the infinitely many constraints with k of them, if those k constraints are suitably chosen. However, both the integer k and the k index points corresponding to the k 'suitable' constraints are not known. Therefore, the k unknown 'suitable' index points are taken as variables for an estimated k which is large enough. This leads to the dual parametrization method which transforms the SIP problem into a finite nonlinear programming problem. It is shown in [7] that for the positive definite case, a global solution of the nonlinear programming problem obtained from the dual parametrization method directly gives rise to the optimal solution of the SIP problem.

Here we will present the dual parametrization results in a general setting, namely in the case where there are multiple ($m > 1$) infinite constraints. Especially, we will prove that the bound for the optimal parametrization number k^* remain to be n for $m > 1$. This result is not surprising because, as far as the total number of constraints involved is concerned, there is no difference between a single infinite constraint and several infinite constraints. Now the question is how to find a global solution to the nonlinear optimization problem—the parametrized dual problem. By using a method similar to that of [7], we can prove that all results obtained for the case $m = 1$ carry over to the case $m > 1$ and the dual parametrization method applies to the general case. Global optimization problem is notoriously difficult to solve. Efficient numerical algorithms for global solution have been possible only for some special classes of optimization problems. In [9], a numberical method for computing a global solution of the parametrized dual problem is developed based on some special feature of the parametrized dual problem. The global solution method further leads to a dual parametrization algorithm for the original SIP problem. The algorithm has a demonstrated efficiency in practical computations. However, the global solution method involves an iterative scheme which requires to solve problems with increasingly large number of constraints. Therefore, some improvement on the dual parametrization algorithm, namely the introduction of an adaptive scheme, is desirable. This desired improvement is considered in this paper. We note that the dual parametrization method is different from the discretization methods existing for SIP problems. Discretization methods usually replace the infinitely many constraints with a finite number of constraints. By solving the corresponding finite optimization problem (or a sequence of them), an approximate solution, which is usually infeasible, is obtained. The dual parametrization method works with the dual problem and tries to find an approximate solution of it. Once this is done, one more local search will lead to the exact solution (up to computer precision) of the SIP problem.

We will first present the dual parametrization method for problem (P) in Section 2. The theory of dual parametrization method for the multiple infinite constraint case is similar to the single infinite constraint case. Therefore, most results will be stated without proof. Following Section 2, an adaptive dual parametrization algorithm will be developed in Section 3. Convergence result regarding the global optimization of the parametrized dual is given. The efficiency of the adaptive algorithm is shown by solving a number of numerical examples.

2. Dual parametrization

In this section we extend the main results in dual parametrization technique to the case where multiple infinite constraints are considered. For the remainder of this paper, we assume that the following constraint qualification is satisfied.

ASSUMPTION 1 (Slater Condition). There is an $x_0 \in R^n$ such that

$$A(y)x_0 - b(y) < 0 \quad \text{for all } y \in Y . \tag{3}$$

We denote by $C(Y)$ the Banach space of all continuous real functions on Y equipped with the supremum norm, and by $M(Y)$ the space of all signed finite regular Borel measures on Y. It is known that $M(Y)$ is the dual space of $C(Y)$. Let V be the cone of $C(Y)$ consisting of all the nonnegative functions in $C(Y)$. The cone in $M(Y)$ associated with V, denoted by V', consists of all the nonnegative elements (nonnegative as measure) of $M(Y)$. Thus, $\Lambda \in V'$ if and only if $\Lambda(f) \geqslant 0$ for all $f \in V$. We will use the same sample '$\geqslant$' to denote the partial orders in both $C(Y)$ and $M(Y)$ induced by V and V', respectively. To be more specific, if f and g are two elements in $C(Y)$ (respectively, $M(Y)$), we write $f \geqslant g$ (equivalently, $g \leqslant f$) if and only if $f - g \in V$ (respectively, V').

Let $A: R^n \to C(Y)$ be the operator defined by

$$(Ax)(y) = A(y)x \quad \text{for } y \in Y \tag{4}$$

and denote by A^* the dual of A. Note that we have used the same symbol A for both the matrix function and the corresponding operator. However, this should not cause any confusion in the context. Using the above notations, problem (P) can be stated as

$$\begin{aligned} &\min \quad f(x) = (1/2)x^TQx + p^Tx , \\ &\text{subject to} \quad Ax - b \leqslant 0 . \end{aligned}$$

The Dorn's dual of problem (P) can be written as:

PROBLEM (P$'$)

$$\begin{aligned} &\min_{x,\Lambda} \; (1/2)x^TQx + \langle \Lambda, b \rangle \\ &\text{s.t.} \quad Qx + p + A^*\Lambda = 0 \\ &\qquad\;\; \Lambda \geqslant 0 \end{aligned} \tag{5}$$

where

$$\langle \Lambda, b \rangle = \int_Y b(y)\, \mathrm{d}\Lambda(y)\,. \tag{6}$$

The following three preliminary results, including the KKT optimality conditions and the Caratheodory's lemma, are important in the proof of the main result. The first two results are standard (see, e.g., [7, 11, 12] and the third one is given in [7]. We state them in the following without proof.

LEMMA 2.1 (KKT conditions). *Let the Slater constraint qualification be satisfied. The minimum of problem* (P) *is achieved at* $x^* \in R^n$ *if and only if* x^* *is feasible and three exists a* $\Lambda^* \in M(Y)$ *such that*

$$\begin{aligned} &Qx^* + p + A^*\Lambda^* = 0\,, \\ &\langle \Lambda^*, Ax^* - b \rangle = 0\,, \\ &\Lambda^* \geqslant 0 \end{aligned} \tag{7}$$

LEMMA 2.2 (Carathéodory). *Let* X *be a subset of* R^n. *If* $x \in coneX$, *i.e.,* x *is a nonnegative linear combination of points in* X, *then there exist* n *numbers* $\alpha_i \geqslant 0$ *such that*

$$x = \sum_{i=1}^{n} \lambda_i x^i$$

for some $x^i \in X$, $i = 1, 2, \ldots, n$, *i.e.,* x *can be represented as a nonnegative linear combination of at most* n *points of* X.

LEMMA 2.3. *Let Assumptions 1 be satisfied, and assume that the minimum of problem* (P) *is achieved at* $x^* \in R^n$. *Then* Λ^* *is a multiplier satisfying the KKT conditions (7) if and only if* (x^*, Λ^*) *is a solutions to the dual problem* (P').

The dual parametrization method is based on the following result.

THEOREM 2.1. *Let Assumption 1 be satisfied, and assume that the minimum of problem* (P) *is achieved at* $x^* \in R^n$. *Then the solution set of the dual problem* (P') *contains a solution pair of which the measure has a finite support of no more than* n *points*.

Proof. In view of Lemma 2.3, we need only to prove that there is a finite support measure Λ^* with no more than n supporting points, such that (x^*, Λ^*) satisfies the KKT conditions (7).

Since x^* is a primal solution, from the KKT theorem, there exists a measure $\bar{\Lambda}$ such that $(x^*, \bar{\Lambda})$ satisfies the KKT conditions (7). Denote the active set of problem (P) at x^* by $Y(x^*)$:

$$Y(x^*) = \{y \in Y\colon A(y)x^* - b(y) = 0\}\,. \tag{8}$$

From the continuity of $A(y)$ and $b(y)$ and the compactness of Y, we see that $Y(x^*)$ is either compact or empty. It follows from (7) that the support of $\bar{\Lambda}$ satisfies

$$\operatorname{supp}(\bar{\Lambda}) \subset Y(x^*) . \tag{9}$$

Thus, (7) means

$$Qx^* + p + \int_{Y(x^*)} A(y)T \, d\bar{\Lambda} = 0 . \tag{10}$$

Hence

$$-(Qx^* + p) = \int_{Y(x^*)} A(y)^T \, d\bar{\Lambda} \in \operatorname{cone}\{a^i(y): y \in Y(x^*), \qquad i = 1, 2, \ldots, m\} . \tag{11}$$

where $a^i(y)$ is the ith column of $A(y)^T$. From Lemma 2.2, there exist an integer $k \leqslant n$, k vectors $a^{i_j}(y_j^*)$, $j = 1, 2, \ldots, k$, and k constants $\alpha_j^* \geqslant 0$, $j = 1, 2, \ldots, k$ such that

$$-(Qx^* + p) = \sum_{j=1}^{k} \alpha_j^* a^{i_j}(y_j^*) \tag{12}$$

where $1 \leqslant i_j \leqslant m$ and $y_j^* \in Y(x^*)$, $j = 1, 2, \ldots, k$. Let λ_j^* be the m-dimensional vector whose elements are all zero except for its i_j-th element which is α_j^*. Define an m-dimensional measure Λ^* with a finite support $\{y_j^*: j = 1, 2, \ldots, k\}$ by

$$\Lambda(\{y_j^*\}) = \lambda_j^*, \quad j = 1, 2, \ldots, k . \tag{13}$$

The measure Λ^* thus defined has a finite support of $k \leqslant n$ supporting points and satisfies $\Lambda^* \geqslant 0$,

$$\begin{aligned} Qx^* + p + \int_Y A(y)^T \, d\Lambda^*(y) &= Qx^* + p + \sum_{j=1}^{k} A(y_j^*)^T \lambda_j^* \\ &= Qx^* + p + \sum_{j=1}^{k} \alpha_j^* a^{i_j}(y_j^*) \\ &= 0 \end{aligned} \tag{14}$$

and, since $y_j^* \in Y(x^*)$,

$$\begin{aligned} \langle \Lambda^*, Ax^* - b \rangle &= \sum_{j=1}^{k} (A(y_j^*)x^* - b(y_j^*))^T \lambda_j^* \\ &= 0 . \end{aligned} \tag{15}$$

Thus (x^*, Λ^*) satisfies the KKT conditions.

A question one may ask is how to identify the integer k. In fact, identifying k is not easy before knowing the solution of problem (P'_k). Fortunately, that is not important as our algorithm to be presented later will require k increase as the iteration goes on. However, a result about this parametrization number given in [9] is helpful in understanding the possible values of k. The result is given below without proof. For the proof, we refer interested readers to [9].

LEMMA 2.4. *For problem* (P'_k), *the following are true.*

(a) The optimal value sequence $\{v(P'_k)\}$ *is non-increasing and there is a* $k^* \geqslant 0$ *such that*

$$v(P'_{k^*}) = v(P'_k), \quad \text{for all } k \geqslant k^*$$

and if $k^* \geqslant 1$

$$v(P'_{k^*-1}) > v(P'_{k^*}) .$$

(b) The number k^* *in (a) is the minimum integer such that for* $k \geqslant k^*$, *a global solution of problem* (P'_k) *provides the solution of problem (P) in the sense that if* (x^*, λ^*, τ^*) *is a global solution of problem* (P'_k), *then* x^* *is the solution of problem (P).*

(c) The number k^* *satisfies* $0 \leqslant k^* \leqslant n$.

(d) If $0 \leqslant k < k^*$, *then*

$$v(P'_k) > v(P'_{k+1}) . \tag{16}$$

The number k^* in the above lemma is called the minimum parametrization number in [9]. The importance of Theorem 2.1 lies in the fact that it allows us to reduce problem (P′) to a finite dimensional problem. In order to solve the primal problem (P), we need only to find a solution pair (x^*, Λ^*) of problem (P′). Thus, from Theorem 2.1, we can restrict our search for Λ^* to those nonnegative measures having a finite support of no more than k supporting points, assuming $k \geqslant k^*$. Such a measure, denoted by Λ_k, is characterized by its k supporting points $y_i \in Y$, $i = 1, 2, \ldots, k$, and the corresponding measure $\lambda_i = \Lambda_k(\{y_i\}) \geqslant 0$, $i = 1, 2, \ldots, k$ at each point. Replacing Λ in problem (P′) by Λ_k, we obtain the following finite dimensional mathematical programming problem

PROBLEM (P'_k)

$$\min_{x,\lambda,\tau} \ (1/2)x^T Qx + \sum_{i=1}^{k} b(y_i)^T \lambda_i$$

$$\text{s.t.} \quad Qx + p + \sum_{i=1}^{k} A(y_i)^T \lambda i = 0 \tag{17}$$

$$\lambda_i \geqslant 0, \quad i = 1, 2, \ldots, k ,$$

$$y_i \in Y, \quad i = 1, 2, \ldots, k$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is in the product space $\Pi_{i=1}^{k} R^m$ and $\tau = (y_1, y_2, \ldots, y_k)$ is in the product space $\Pi_{i=1}^{k} R^s$.

Problem (P'_k) is called the parametrized dual of problem (P) ([9]). From the above discussion, we see that for any $k \geqslant k^*$, once a global solution (x^*, λ^*, τ^*) of problem (P'_k) is obtained, then x^* must be the solution of problem (P). Thus, in order to sovle problem (P), we need only to deal with problem (P'_k). In the next section, we present a numerical algorithm for solving problem (P'_k) for some $k \geqslant k^*$.

3. Algorithm

In the previous section, we have reduced problem (P′) to a finite programming problem. Therefore, in order to solve problem (P), we need only to solve problem (P'_k). However, problem (P'_k) is nonlinear and the challenge is to find its global solution. In [9], an algorithm is proposed to compute a global solution of problem (P'_k). The algorithm combines a refinement scheme on the parametrization of the index set Y and a local search. It can be described briefly as follows: Choose an integer k and a k tuple $\tau = (y_1, y_2, \ldots, y_k)$ of index points $y_i \in Y$, $i = 1, 2 \ldots, k$. Solve the following problem.

PROBLEM ($P'_k(\tau)$)

$$\min_{x,\lambda} (1/2)x^T Qx + \sum_{i+1}^{k} b(y_i)^T \lambda_i \tag{18}$$

$$\text{s.t.} \quad Qx + p + \sum_{i+1}^{k} A(y_i)^T \lambda_i = 0 \tag{19}$$

$$\lambda_i \geqslant 0, \quad i = 1, 2, \ldots, k\,. \tag{20}$$

At the optimal solution, increase k and refine the grid points $y_1, y_2, \ldots, y_k$ in such a way that $\max_{y \in Y} \min_{1 \leqslant i \leqslant k} \|y_i - y\| \to 0$ as $k \to \infty$. The solve problem ($P'_k(\tau)$) again. Repeat this procedure until some stopping criterion is satisfied. At the termination, a vector of approximate index points $\bar{\tau}$ together with the corresponding optimal solution $(\bar{x}, \bar{\lambda})$ to problem ($P'_k(\bar{\tau})$) is obtained. The triple $(\bar{\tau}, \bar{x}, \bar{\lambda})$ provides an approximate solution to problem (P'_k). If this approximate solution is good enough, it is expected that $(\bar{\tau}, \bar{x}, \bar{\lambda})$ lies in the basin of a global solution of problem (P'_k). Then a global solution for problem (P'_k) can be obtained by performing a local search starting from this approximate solution, provided that the technique used in the local search is stable in the sense that it always finds the local solution which shares the same basin with the starting point (initial solution). One of the advantages of the above-mentioned algorithm is that for each fixed τ, problem ($P'_k(\tau)$) is a convex quadratic problem with linear constraints and is easy to solve. However, when k increases, the number of constraints in problem ($P'_k(\tau)$) becomes very large. This will increase the computing cost and may cause numerical difficulties. In the following, we propose an adaptive scheme so that at each iteration, a much smaller number of constraints in problem ($\mathrm{P}'_k(\tau)$) are selected.

LEMMA 3.1. *Let* $\tau = (y_1, y_2, \dots, y_k) \in \Pi_{j=1}^k R^s$ *be any k tuple of index points. Then, problem* ($P'_k(\tau)$) *is the Dorn's dual of*

PROBLEM ($\mathrm{P}_k(\tau)$)

$$\min_x (1/2)x^T Qx + p^T x \tag{21}$$

$$\text{s.t.} \quad A(y_j)x - b(y_i) \leqslant 0, \quad j = 1, 2, \dots, l_i\,. \tag{22}$$

Furthermore, a vector $\hat{x}$ *is the solution of problem* ($P_k(\tau)$) *iff there exists some* $\hat{\lambda} = (\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_k) \in \Pi_{j=1}^k R^m$ *such that* $\{(\hat{x}, \hat{\lambda})\}$ *is a solution of problem* ($P'_k(\tau)$). *The optimal values of problem* ($P_k(\tau)$) *and problem* ($P'_k(\tau)$) *satisfies*

$$v((P_k(\tau)) = -v(P'_k(\tau)).$$

Proof. The first part of the lemma is easy to check since this is a special case of the primal-dual relation of problem (P) and problem (P′). the second part of the lemma is standard and can be found in [8]. □

Let $\{k_i\}_{i=1}^\infty$ be a given sequence of integers. For $i \geqslant 1$, let $Y_i = \{y_j^i\colon j = 1, 2, \dots, k_i\}$ be a given subset of Y satisfying

$$d(Y_i, Y) \hat{=} \max_{y \in Y} \min_{1 \leqslant j \leqslant k_i} |y - y_j^i| \to 0\,. \tag{23}$$

We propose the following algorithm:

ALGORITHM I

1. Choose an arbitrary x^0, a small number $\varepsilon < 0$, an integer N, and a sequence of parametrization sets

   $$Y_i = \{y_j^i\colon j = 1, 2, \dots, k_i\},\ i = 0, 1, \dots$$

   satisfying (23).
2. Let $E^0 = \phi$. Set $i = 0$.
3. Set $i = i + 1$,

   $$V^i = \{y \in Y_i | A(y)x^{i-1} - b(y) \geqslant 0\} \cup E^{i-1}.$$

   Suppose V^i has l_i elements $V^i = \{z_1^i, z_2^i, \dots, z_k^i\}$. Let $\tau^i = (z_1^i, z_2^i, \dots, z_{l_i}^i)$.
4. Solve problem ($\mathrm{P}'_{l_i}(\tau^i)$) to obtain a solution ($x^i$, λ^i).
5. If $i \leqslant N$ or $f(x^i) - f(x^{i-1}) \geqslant \varepsilon$, let

   $$E^i = \{y \in V^i | A(y)x^i - b(y) = 0\}\,,$$

   go to step 3.

6. Solve problem $(P'_{l_i}(\tau^i))$, starting from (x^i, λ^i, τ^i). The optimal solution is denoted by (x^*, λ^*, τ^*). Then x^* is taken as the solution for problem (P).

In the following, we prove that if the sequence of parametrization points Y_i, $i = 1, 2, \ldots$, satisfies (23), then the solution sequence $\{x^i\}$ obtained from Algorithm I converges to the solution of the problem (P).

THEOREM 3.1. *If (23) is satisfied, then the sequence* $\{x^i\}$ *obtained from Algorithm I converges to the solution of problem* (*P*). *Therefore, if* ε *and N are suitably chosen, the* x^* *obtained in Step 6 is the optimal solution of problem* (*P*).

Proof. From Lemma 3.1, we see that x^i is the solution of problem $(P_{l_i}(\tau^i))$. Since E^i is the active index set of problem $(P_{l_i}(\tau^i))$ at the solution x^i, if we reduce the constraints of problem $(P_{l_i}(\tau^i))$ to those corresponding to E^i, the solution f the problem is still x^i. On the other hand, x^{i+1} is the solution of problem $(P_{l_{i+1}}(\tau^{i+1}))$ of which the constraint index set is V^{i+1} which contains E^i as a subset. Thus it is easy to see that

$$f(x_i) \leqslant f(x^{i+1}), \quad i + 1, 2, \ldots . \tag{24}$$

The existence of a Slater point x_0 for problem (P) shows that the sequence $\{f(x_i)\}$ is bounded from above by $f(x_0)$. Thus, there exists some constant f^* such that

$$f(x^i) \to f^* \; (i \to \infty) . \tag{25}$$

The strict convexity of $f(x)$ and the boundedness of $\{f(x^i)\}$ shows that the sequence $\{x^i\}$ is bounded. Let $\bar{x}$ be a limit point of $\{x^i\}$. Then there exists a subsequence $\{x_{i_k}\}$ of $\{x^i\}$ such that

$$\lim_{k\to\infty} x_{i_k} = \bar{x} . \tag{26}$$

We now show that $\bar{x}$ is a feasible point of problem (P). In fact, if $\bar{x}$ is not a feasible point of problem (P), then there exists $y_0 \in Y$ such that $g(t_0, \bar{x}) > 0$. Let $g(y_0, \bar{x}) = 2\delta$. Since $g(y, x)$ is continuous, we see that there exists $\varepsilon > 0$ such that

$$|g(y, x) - g(y_0, \bar{x})| < \delta, \quad \text{for } \|y - y_0\| < \varepsilon, \|x - \bar{x}\| < \varepsilon \tag{27}$$

as a result, we have

$$g(y, x) \geqslant \delta, \quad \text{for } \|y - y_0\| < \varepsilon, \|x - \bar{x}\| < \varepsilon . \tag{28}$$

From (23) and (26), thee exists an integer K such that for $k \geqslant K$, we can choose $y^{i_k}_{j_k} \in Y_{i_k}$ satisfying

$$\|y^{i_k}_{j_k} - y_0\| < \varepsilon, \|x^{i_k} - \bar{x}\| < \epsilon/2, \quad \text{for } k \geqslant K . \tag{29}$$

Thus,

$$g(y^{i_k}_{j_k}, x^{i_k}) \geqslant \delta, \quad \text{for } k \geqslant K . \tag{30}$$

It is clear that $y^{i_k}_{j_k} \in V^{k+1}$ and hence

$$g(y^{i_k}_{j_k}, x^{i_k+1}) \leqslant 0, \quad \text{for } k \geqslant K . \tag{31}$$

Since for $k \geqslant K$, $\|y^{i_k}_{j_k} - y^0\| < \varepsilon$ and

$$|g((y^{i_k}_{j_k}, x^{i_k+1}) - g(y_0, \bar{x})| \geqslant 2\delta ,$$

(27) shows that

$$\|x^{i_k+1} - \bar{x}\| \geqslant \varepsilon \quad \text{for } k \geqslant K . \tag{32}$$

From (29) and (32), we have

$$\|x^{i_k+1} - x^{i_k}\| \geqslant \varepsilon/2, \quad \text{for } k \geqslant K . \tag{33}$$

It is clear that $\{x^{i_k+1}\}$ has a converging subsequence. Without loss of generality, we suppose that the subsequence converges itself:

$$x^{i_k+1} \to \hat{x}, \ (k \to \infty) .$$

Then we have

$$\|\bar{x} - \hat{x}\| \geqslant \varepsilon/2 , \tag{34}$$

$$f(\bar{x}) = f(\hat{x}) . \tag{35}$$

Note that x^{i_k} is the solution of problem

$$\min_x \ (1/2)x^TQx + p^Tx \tag{36}$$

$$\text{s.t.} \quad A(y)x - b(y) \leqslant 0, \quad y \in E^{i_k} . \tag{37}$$

and x^{i_k+1} is a feasible point of the same problem. Thus, since the feasible set of the above problem is convex and its objective function is strictly convex, we see that $f(x)$ is strictly monotone along the segment connecting x^{i_k} and x^{i_k+1}. Thus,

$$f(x^{i_k}) < f((x^{i_k} + x^{i_k+1})/2) < f(x^{i_k+1}) . \tag{38}$$

Let $k \to \infty$ in (38) inequality, we obtain

$$f(\bar{x}) \leqslant f((\bar{x} + \hat{x})/2) \leqslant f(\hat{x}) . \tag{39}$$

The strict convexity of $f(x)$, together with (35) and (39), shows that $\bar{x} = \hat{x}$ which contradicts to (34). Therefore, $\bar{x}$ is feasible to problem (P).

Next we show that the whole sequence $\{x^i\}$ converges to the solution x^* of problem (p). Suppose $\{x^i\}$ does not converge. Then there two subsequences $\{x^{i_k}\}$ and $\{x^{j_k}\}$ converging to x' and x'', respectively, where $x' \neq x''$. Then both x' and x'' are feasible to problem (P) as we proved above. The point $(x' + x'')/2$ is feasible to problem (P) and hence feasible to problem $(P_{l_{i_k}}(\tau))$ for all $k \geqslant 1$. Therefore,

$$f((x' + x'')/2) < (f(x') + f(x''))/2 = f^*.$$

Since $f(x^{i_k}) \to f^*)$ as $k \to \infty$, we have, for sufficiently large k,

$$f((x' + x'')/2) < f(x^{i_k}).$$

This contradicts to the fact that x^{i_k} is the solution to problem $(\mathrm{P}_{l_{i_k}}(\tau^{i_k}))$. Theorem, $\{x^i\}$ converges to x^*. It is clear that x^* is the solution of problem (P). The problem is complete. □

4. Numerical example

In this section, we demonstrate the efficiency of the adaptive algorithm by solving two numerical examples. The two examples are from previous works and were solved in [9]. For these examples, the index sets for the infinite constraints are intervals of R^1. The sequence of parametrization sets Y_i, $i = 0, 1, \ldots,$ are chosen in the following way. Once Y_0 is given, Y_k, $k \geqslant 1$ consists of the midpoints of each pair of adjacent points in $\cup_{j=1}^{k-1} Y_j$.

EXAMPLE 1.

$$\min\ x^T Q x$$
$$\text{s.t.}\quad A(y)x - b(y) \leqslant 0, \quad \text{for } y \in [0, 5]$$

where

$$Q = \begin{bmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 4 \end{bmatrix} \tag{40}$$

$$A(y) = (-e^{-(y-1(5/16))^2}, -e^{-(y-2(5/16))^2}, \ldots, -e^{-(y-16(5/16))^2}) \tag{41}$$

$$b(y) = -3 - 4.5 \sin(4.7\pi(y - 1.23)/8)\,. \tag{42}$$

The original form of this example is given in [6]. It was modified into the current form in [9]. In order to solve this problem using the adaptive algorithm developed in the previous section, we choose the parametrization sequence $\{Y_i\}$ as follows: $Y_1 = \{0, 2.5, 5\}$, for $i \geqslant 2$, Y_i consists of the midpoints of all pairs of adjacent points in $\cup_{j=1}^{i-1} Y_j$. Computational results are shown in Table 1. We note that the active constraint index points at the solution is identified as $y_1^* = 2.06165$ and $y_2{}^* = 5.00000$ which is exactly the same as those obtained in [6] and [9]. The optimal solution also coincides with the previous computations up to seven decimal digits. We tested several different values for x^0. All results are similar to that presented in Table 1.

EXAMPLE 2. The one-sided L^2 approximation of the tangent function on [0, 1) by polynomials of degree not exceeding n is stated as [7]

Table 1. Results for Example 1

Approximate x	[0.084044	0.142695	0.258183	0.436827	
	0.707666	1.046537	1.345125	1.453519	
	1.303190	0.987357	0.708656	0.664137	
	0.914220	1.451353	1.763188	2.606933]	
Approximate λ	0.000000	1.975311	16.463732	0.000000	24.350055
Optimal x	[0.076461	0.204917	0.486116	0.927634	
	1.437805	1.808159	1.845673	1.530850	
	1.039785	0.610216	0.404887	0.495781	
	0.855648	1.458993	1.798618	2.668698]	
Optimal objective	154.116154				

$$\min_{x} \int_0^1 \left(\sum_1^n x_i t^{i-1} - \tan(t) \right)^2 \mathrm{d}t$$

$$\text{s.t.} \quad \sum_1^n - x_i t^{i-1} \leqslant - \tan(t), \quad \text{for } t \in [0, 1] .$$

The problem is transformed into the standard quadratic form (the form of problem (P)) by specifying Q, p, A and b.

$$Q = [2/(i+j-1)]_{n \times n} ,$$

$$p = \int_0^1 [1, t, \ldots, t^{n-1}]^T \tan(t) \, \mathrm{d}t ,$$

$$A(t) = [-1, -t, \ldots, -t^{n-1}] ,$$

$$b(t) = -\tan(t) .$$

As in [9], we solved this problem for the cases from $n = 2$ to $n = 9$. The numerical solutions obtained by the adaptive algorithm are the same as those presented in [9]. It is observed that, for the cases $n \geqslant 6$, the adaptive algorithm is faster than the algorihm given in [9]. This suggest that the adaptive algorithm of this paper is more efficient for larger problems. The sequence of parametrization sets are determined as in Example 1 except we choose $Y_0 = \{0, 1\}$ here. The computational results for the case of $n = 9$ is presented in Table 2. The computed optimal solution and the active

Table 2. Results for Example 2

n	9				
Approximate x	[0.002584	0.920014	0.607620	− 1.334997	
	1.517024	0.482381	− 0.286100	− 1.490102	1.060138]
Optimal x	[0.002560	0.919760	0.606718	− 1.335399	
	1.516566	0.484284	− 0.285038	− 1.410207	1.059893]
Optimal objective	0.278718				

constraint index points at the computed primal solution are the same as those obtained in [9].

5. Comments

In this paper, we developed a global optimization algorithm for the class of nonlinear programming problems obtained from positive LQ SIP problems by the dual parametrization method. Convergence result is given and some useful properties regarding the relation between the primal SIP and transformed nonlinear programming problem are discussed. Two existing examples are solved by the algorithm and the numerical results show that the algorithm is efficient in finding the global solution.

References

[1] Fiacco, A.V. and Ishizuka, Y. (1990), Suggested Research Topics in Sensitivity and Stability Analysis for Semi-infinite Programming, *Annals Oper. Res.* 27, 65–76.
[2] Goberna M.A. and M.A. Lopez (1998), *Linear Semi-infinite Optimization*, John Wiley & Sons, Chichester.
[3] Gustafson S.-Å. and Kortanek, K.O. (1983), Semi-infinite Programming and Applications, in Bachen, A., et al. (eds.), *Mathematical Programming: the State of the Art*, Springer, Berlin, pp. 138–157.
[4] Hettich, R. (1983), A review of Numerical Methods for Semi-infinite Optimization, in Fiacco, A.V. and Kortanek, K.O. (eds.), *Semi-infinite Programming and Applications*, Springer, Berlin, pp. 158–178.
[5] Hettich R. and Kortanek, K.O. (1993), Semi-infinite Programming: Theory, Method, and Applications, *SIAM Review* 35, 380–429.
[6] Liu, Y. Teo, K.L. and Ito, S. (1999), A Dual Parameterization Approach to Linear-quadratic Semi-infinite Programming Problems, *Optimization Methods and Software*, 10, 471–491.
[7] Ito, S. Liu Y. and Teo, K.L. A Dual Parametrization Method for Convex Semi-infinite Programming, *Annals of Operations Research*, to appear.
[8] Liu Y., Teo, K.L. and Ito, S. (1999), A Dual Parametrization Approach to Linear-Quadratic Semi-infinite Programming Problems, *Optimization Methods and Software* 10, 471–491.
[9] Liu Y., Teo, K.L. and Ito, S. Global Optimization in Linear-Quadratic Semi-infinite Programming, *Computing*, to appear.
[10] Luenberger, D.G. (1969), *Optimization by Vector Space Method*, John Wiley & Sons Inc., New York, London, Sydney, Toronto.
[11] Mangasarian O.L. and Ponstein, J. (1965), Minmax and Duality in Nonlinear Programming, *Journal of Mathematical Analysis and Applications* 11, 504–518.
[12] Rockafellar, R.T. (1970), *Convex Analysis*, *Princeton Mathematical Series 28*, Princeton University Press, Princeton, NJ.
[13] Teo, K.L. Rehbock, V. and Jennings, L.S. A New Computational Algorithm for Functional Inequality Constrained Optimization Problems, *Automatica*, 789–792.